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DYNAMIC STRESS INTENSITY FACTORS AROUND TWO PARALLEL CRACKS IN AN INFINITE–ORTHOTROPIC PLANE SUBJECTED TO INCIDENT HARMONIC STRESS WAVES

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Abstract—The time-harmonic problem of determining stresses around two parallel cracks in an infinite orthotropic plane is studied. Incident stress waves impinge on the two cracks normal to their surfaces. The Fourier transform technique is used to reduce the boundary conditions to four simultaneous integral equations which are then solved by expanding the differences in the crack surface displacements in a series. The unknown coefficients in the series are calculated using the Schmidt method. Numerical calculations are carried out for the dynamic stress intensity factors in a boron–epoxy composite material, a carbon fiber reinforced plastic, a modulite II graphite-epoxy composite and an isotropic material.

1. INTRODUCTION

The dynamic problem for a finite crack was first investigated by Loeber and Sih (1968). They obtained the stress intensity factors around a crack during the passage of a time-harmonic antiplane shear wave. Later, they also solved the crack problem for a compression wave and a vertically polarized shear wave (Sih and Loeber, 1969). Using a somewhat different approach, Mal (1970) treated the same problems independently. Following these pioneering works, solutions have been obtained for numerous time-harmonic problems as well as transient problems, as shown in the recently published *Stress Intensity Factors Handbook* (Murakami, 1987). According to the handbook, the peak value of the dynamic stress intensity factor K_i^{peak} is generally about 1.20–1.60 times as large as that of the corresponding static value K_i^{static}. However, two cases exist in which the K_i^{peak}/K_i^{static} ratio takes on a considerably larger value. These are (i) the time-harmonic problem for a finite crack in a half-plane where the plane surface is parallel to the crack (Keer *et al.*, 1984) and (ii) the time-harmonic problem for two parallel cracks in an infinite plane subjected to waves that impinge perpendicular to the cracks (Takakuda, 1982). For the former, the $K_{\rm I}^{peak}/K_{\rm I}^{static}$ ratio is 4.12 for $d/a = 0.8$, with $2a$ being the crack length and d being the depth beneath the free surface. For the latter, the ratio is 4.16 for $h/a = 1.0$, with h being the distance between the two parallel cracks.

Recently, composite materials, which are essentially orthotropic materials, have attracted attention due to their high strength and relative lightness. However, these materials can be weakened by cracks that appear at the interfaces between the fiber and the matrix. Consequently, cracked composite materials can be loaded dynamically. These dynamic stresses in an infinite orthotropic medium weakened by a crack were determined by Ohyoshi (1973) for the time-harmonic problem, and by Kassir and Bandyopadhyay (1983) for the transient problem. In these solutions, the K_i^{peak}/K_i^{static} ratio falls between 1.20 and 1.30.

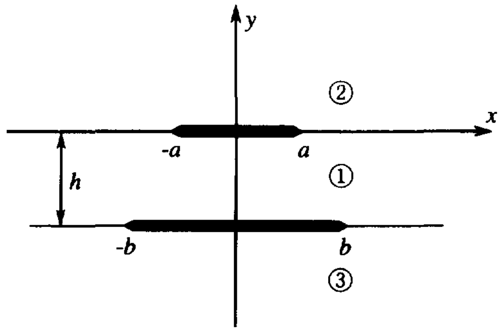

Fig. 1. Geometry and coordinate system.

As mentioned above, the peak value of the dynamic stress intensity factors around a crack in a half-plane is very large for cases in which the crack is situated parallel to the stress-free surface. However, this problem may not be realistic because external tensile force rarely acts on a plane surface. Large K_I^{peak}/K_I^{static} ratios are also seen in two parallel cracks during the passage of time-harmonic stress waves, although the ratio for the corresponding transient problem is only about 1.3 (Itou, 1995). Cracks ordinarily occur with these surfaces intersecting perpendicular to the line along which the external force acts. Therefore, it is of practical interest to solve the problem for the stress intensity factors around two parallel cracks in an orthotropic plane subjected to time-harmonic stress waves so as to avoid catastrophic fractures in composite materials.

In the present paper, dynamic stresses have been determined in an infinite orthotropic plane containing two parallel cracks. Time-harmonic stresses pass perpendicularly through the material to the surface of each crack. The Fourier transform technique is used to reduce the mixed boundary value conditions to a set of dual integral equations. The equations are solved by expanding the differences in the crack surface displacements in a series. The unknown coefficients in the series are then determined using the Schmidt method (Itou, 1976) and the values of the dynamic stress intensity factors are calculated numerically for several composite materials.

2. FUNDAMENTAL EQUATIONS

Consider a crack located on the x-axis from $-a$ to $+a$, with respect to the rectangular coordinates (x, y), and another along the x-axis from $-b$ to $+b$ at $y = -h$, as shown in Fig. 1. The problem is first formulated for the generalized plane stress condition. For convenience, we refer to $-h \leqq y \leqq 0$ as layer ①, $0 \leqq y$ as upper half-plane ②, and $y \leqq -h$ as lower half-plane ③.

For an orthotropic medium, the equations of motion are given as follows (from Kassir and Bandyopadhyay, 1983),

$$c_{11}\,\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + (1+c_{12})\,\partial^2 v/(\partial x\,\partial y) = (1/c_t^2)\,\partial^2 u/\partial t^2$$
$$c_{22}\,\partial^2 v/\partial y^2 + \partial^2 v/\partial x^2 + (1+c_{12})\,\partial^2 u/(\partial x\,\partial y) = (1/c_t^2)\,\partial^2 v/\partial t^2 \tag{1}$$

with

$$c_t = (\mu_{xy}/\rho)^{1/2}, \quad c_{11} = E_x\{1-(E_y/E_x)\nu_{xy}^2\}/\mu_{xy}$$
$$c_{22} = c_{11}E_y/E_x, \quad c_{12} = \nu_{xy}c_{22} \tag{2}$$

where u and v are defined as the x and y components of the displacement, respectively, E_x and E_y are Young's modulus, μ_{xy} is the modulus of rigidity, ν_{xy} is Poisson's ratio, ρ is the density of the material and t is time.

The stresses are given as

$$\sigma_x/\mu_{xy} = c_{11}\,\partial u/\partial x + c_{12}\,\partial v/\partial y, \quad \sigma_y/\mu_{xy} = c_{12}\,\partial u/\partial x + c_{22}\,\partial v/\partial y,$$
$$\tau_{xy}/\mu_{xy} = \partial u/\partial y + \partial v/\partial x. \tag{3}$$

3. BOUNDARY CONDITIONS

The incident displacement waves are assumed to be

$$u^{(i)} = 0,$$
$$v^{(i)} = v_0 \exp\left[i\{\omega y/(c_t\sqrt{c_{22}}) + \omega t\}\right] \tag{4}$$

where v_0 is a constant and ω is the circular frequency. Substituting eqn (4) into eqn (3) results in

$$\sigma_x^{(i)} = p_0(c_{12}/c_{22}) \exp\left[i\{\omega y/(c_t\sqrt{c_{22}}) + \omega t\}\right]$$
$$\sigma_y^{(i)} = p_0 \exp\left[i\{\omega y/(c_t\sqrt{c_{22}}) + \omega t\}\right]$$
$$\tau_{xy}^{(i)} = 0 \tag{5}$$

with

$$p_0 = v_0 \mu_{xy} c_{22} \omega i/(c_t\sqrt{c_{22}}). \tag{6}$$

For convenience, the time-factor $\exp(i\omega t)$ is dropped hereafter. If only the stress intensity factors are considered, the problem can be solved using the following boundary conditions:

$$\sigma_{y1} = \sigma_{y2}, \quad \tau_{xy1} = \tau_{xy2} \quad \text{at } y = 0, \quad |x| < \infty \tag{7}$$

$$\sigma_{y1} = \sigma_{y3}, \quad \tau_{xy1} = \tau_{xy3} \quad \text{at } y = -h, \quad |x| < \infty \tag{8}$$

$$\sigma_{y1} = \sigma_{y2} = -p_0, \quad \tau_{xy1} = \tau_{xy2} = 0 \quad \text{at } y = 0, \quad |x| < a \tag{9a}$$

$$u_1 = u_2, \quad v_1 = v_2 \quad \text{at } y = 0, \quad |x| > a \tag{9b}$$

$$\sigma_{y1} = \sigma_{y3} = -p_0 \exp\left[-i\{\omega h/(c_t\sqrt{c_{22}})\}\right], \quad \tau_{xy1} = \tau_{xy3} = 0 \quad \text{at } y = -h, \quad |x| < b \tag{10a}$$

$$u_1 = u_3, \quad v_1 = v_3 \quad \text{at } y = -h, \quad |x| > b \tag{10b}$$

where the variables with the subscript $i = 1$ are those for layer ①, $i = 2$ for upper half-plane ②, and $i = 3$ for lower half-plane ③. It is assumed that the faces of the cracks do not come into contact during vibration.

4. ANALYSIS

To find the solution, we introduce the Fourier transforms defined by

$$\bar{f}(\xi) = \int_{-\infty}^{\infty} f(x) \exp(i\xi x)\,dx \tag{11a}$$

$$f(x) = (1/2\pi) \int_{-\infty}^{\infty} \bar{f}(\xi) \exp(-i\xi x)\,d\xi. \tag{11b}$$

Applying eqn (11a) to eqn (1) results in

$$(\mathrm{d}^4/\mathrm{d}y^4 + p\,\mathrm{d}^2/\mathrm{d}y^2 + q)(\bar{u}, \bar{v}) = 0 \tag{12}$$

with

$$p = \xi^2(2c_{12} + c_{12}^2 - c_{11}c_{22})/c_{22} + (1 + c_{22})\omega^2/(c_{22}c_t^2)$$
$$q = (\xi^2 c_{11} - \omega^2/c_t^2)(\xi^2 - \omega^2/c_t^2)/c_{22}. \tag{13}$$

The characteristic equation of eqn (12) is

$$\lambda^4 + p\lambda^2 + q = 0. \tag{14}$$

The four roots of λ_1, λ_2, λ_3, λ_4 have the following different forms depending on the values of p and q.

For $p^2 - 4q \geqq 0$ and $-p + (p^2 - 4q)^{1/2} \geqq 0$

$$\lambda_1 = (\gamma_1/2)^{1/2} \tag{15a}$$

with

$$\gamma_1 = |-p + (p^2 - 4q)^{1/2}|. \tag{15b}$$

For $p^2 - 4q \geqq 0$ and $-p + (p^2 - 4q)^{1/2} < 0$

$$\lambda_1 = (\gamma_1/2)^{1/2} i. \tag{15c}$$

For $p^2 - 4q \geqq 0$ and $-p - (p^2 - 4q)^{1/2} \geqq 0$

$$\lambda_3 = (\gamma_2/2)^{1/2} \tag{16a}$$

with

$$\gamma_2 = |-p - (p^2 - 4q)^{1/2}|. \tag{16b}$$

For $p^2 - 4q \geqq 0$ and $-p - (p^2 - 4q)^{1/2} < 0$

$$\lambda_3 = (\gamma_2/2)^{1/2} i. \tag{16c}$$

For $p^2 - 4q < 0$

$$\lambda_1 = (\gamma_3/2)^{1/2}\{\cos(\theta/2) + i\sin(\theta/2)\}$$
$$\lambda_3 = (\gamma_3/2)^{1/2}\{\cos(\theta/2) - i\sin(\theta/2)\} \tag{17a}$$

with

$$\gamma_3 = 2\sqrt{q}, \quad \theta = \tan^{-1}\{(4q - p^2)^{1/2}/(-p)\}. \tag{17b}$$

For all cases, λ_2 and λ_4 are given as

$$\lambda_2 = -\lambda_1$$
$$\lambda_4 = -\lambda_3. \tag{18}$$

Then, the solutions of eqn (12) have the following forms for layer ①, upper half-plane ②, and lower half-plane ③, respectively:

$$\begin{aligned}\bar{u}_1 &= A_1\cosh(\alpha_1 y)+B_1\cosh(\beta_1 y)+C_1\sinh(\alpha_1 y)+D_1\sinh(\beta_1 y)\\ \bar{v}_1 &= E_1\cosh(\alpha_1 y)+F_1\cosh(\beta_1 y)+G_1\sinh(\alpha_1 y)+H_1\sinh(\beta_1 y)\end{aligned} \tag{19}$$

$$\begin{aligned}\bar{u}_2 &= A_2\exp(\alpha_2 y)+B_2\exp(\beta_2 y)\\ \bar{v}_2 &= C_2\exp(\alpha_2 y)+D_2\exp(\beta_2 y)\end{aligned} \tag{20}$$

$$\begin{aligned}\bar{u}_3 &= A_3\exp(\alpha_3 y)+B_3\exp(\beta_3 y)\\ \bar{v}_3 &= C_3\exp(\alpha_3 y)+D_3\exp(\beta_3 y)\end{aligned} \tag{21}$$

where $A_1, B_1, \ldots, D_3$ are the unknown coefficients, and $\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3$ are

$$\alpha_1 = \lambda_1, \quad \beta_1 = \lambda_3, \quad \alpha_2 = \lambda_2, \quad \beta_2 = \lambda_4, \quad \alpha_3 = \lambda_1, \quad \beta_3 = \lambda_3. \tag{22}$$

Branches of α_2, β_2 for upper half-plane ②, and those of α_3, β_3 for lower half-plane ③ must be selected to satisfy the conditions under which u_2, v_2, u_3, v_3 are zero at an infinite distance from the cracks and that only outgoing waves exist. The results are not affected even if branches of α_1, β_1 for layer ① are replaced by λ_2, λ_4, respectively.

Substituting eqns (19), (20), (21) into eqn (1) in the Fourier domain we obtain

$$\begin{aligned}(A_1, C_1) = im_1(G_1, E_1), \quad (B_1, D_1) = im_1(H_1, F_1), \quad A_2 = im_2C_2\\ B_2 = in_2D_2, \quad A_3 = im_3C_3, \quad B_3 = in_3D_3\end{aligned} \tag{23}$$

with

$$\begin{aligned}m_i &= (1+c_{12})\alpha_i\xi/(\alpha_i^2+\omega^2/c_t^2-\xi^2c_{11})\\ n_i &= (1+c_{12})\beta_i\xi/(\beta_i^2+\omega^2/c_t^2-\xi^2c_{11}).\end{aligned} \tag{24}$$

Therefore, the displacements and stresses are now expressed by the unknowns E_1, F_1, G_1, H_1, C_2, D_2, C_3, D_3. By applying eqns (7) and (8), E_1, F_1, G_1, H_1 can now be represented as

$$\begin{aligned}E_1/m_1 &= (C_2/m_2)f_1(\xi)+(D_2/n_2)f_2(\xi)+(C_3/m_3)f_3(\xi)+(D_3/n_3)f_4(\xi)\\ F_1/n_1 &= (C_2/m_2)f_5(\xi)+(D_2/n_2)f_6(\xi)+(C_3/m_3)f_7(\xi)+(D_3/n_3)f_8(\xi)\\ G_1/m_1 &= (C_2/m_2)f_9(\xi)+(D_2/n_2)f_{10}(\xi)+(C_3/m_3)f_{11}(\xi)+D_3/n_3)f_{12}(\xi)\\ H_1/n_1 &= (C_2/m_2)f_{13}(\xi)+(D_2/n_2)f_{14}(\xi)+(C_3/m_3)f_{15}(\xi)+(D_3/n_3)f_{16}(\xi)\end{aligned} \tag{25}$$

where $f_1(\xi), f_2(\xi), \ldots, f_{16}(\xi)$ are given in Appendix A. Equations (23) and (25) show that the Fourier-transformed stresses and displacements can be expressed by coefficients C_2, D_2, C_3, D_3.

Here, the remaining boundary conditions are given by eqns (9a), (9b), (10a), (10b). To satisfy eqns (9b) and (10b), we expand the differences in the displacements at $y = 0$ as follows

$$\begin{aligned}\pi(u_{1a}-u_{2a}) &= \sum_{n=1}^{\infty} a_n \sin\{2n\sin^{-1}(x/a)\} \quad (0 \leqq |x| < a)\\ &= 0 \quad (a < |x| < \infty)\end{aligned} \tag{26}$$

$$\pi(v_{1a}-v_{2a}) = \sum_{n=1}^{\infty} b_n \cos\{(2n-1)\sin^{-1}(x/a)\} \quad (0 \leqq |x| < a)$$

$$= 0 \quad (a < |x| < \infty) \tag{27}$$

and those at $y = -h$ by

$$\pi(u_{1b}-u_{3b}) = \sum_{n=1}^{\infty} c_n \sin\{2n\sin^{-1}(x/b)\} \quad (0 \leqq |x| < b)$$

$$= 0 \quad (b < |x| < \infty) \tag{28}$$

$$\pi(v_{1b}-v_{3b}) = \sum_{n=1}^{\infty} d_n \cos\{(2n-1)\sin^{-1}(x/b)\} \quad (0 \leqq |x| < b)$$

$$= 0 \quad (b < |x| < \infty) \tag{29}$$

where a_n, b_n, c_n, d_n are the unknown coefficients to be determined, and the subscripts a and b are the values at $y = 0$ and $y = -h$, respectively. The Fourier transforms of eqns (26), (27), (28), (29) are

$$(-i)(\bar{u}_{1a}-\bar{u}_{2a}) = \sum_{n=1}^{\infty} a_n\{2n/\xi\}J_{2n}(a\xi)$$

$$(\bar{v}_{1a}-\bar{v}_{2a}) = \sum_{n=1}^{\infty} b_n\{(2n-1)/\xi\}J_{(2n-1)}(a\xi)$$

$$(-i)(\bar{u}_{1b}-\bar{u}_{3b}) = \sum_{n=1}^{\infty} c_n\{2n/\xi\}J_{2n}(b\xi)$$

$$(\bar{v}_{1b}-\bar{v}_{3b}) = \sum_{n=1}^{\infty} d_n\{(2n-1)/\xi\}J_{(2n-1)}(b\xi) \tag{30}$$

where $J_n(\xi)$ is the Bessel function.

The left-hand sides in eqn (30) can now be expressed as

$$(-i)(\bar{u}_{1a}-\bar{u}_{2a}) = (-i)(iG_1M_1/m_1+iH_1N_1/n_1-iC_2M_2/m_2-iD_2N_2/n_2)$$

$$(\bar{v}_{1a}-\bar{v}_{2a}) = F_1+H_1-C_2-D_2$$

$$(-i)(\bar{u}_{1b}-\bar{u}_{3b}) = (-i)\{iE_1M_1\sinh(-\alpha_1h)/m_1+iF_1N_1\sinh(-\beta_1h)/n_1$$

$$+iG_1M_1\cosh(-\alpha_1h)/m_1+iH_1N_1\cosh(-\beta_1h)/n_1$$

$$-iC_3M_3\exp(-\alpha_3h)/m_3-iD_3N_3\exp(-\beta_3h)/n_3\}$$

$$(\bar{v}_{1b}-\bar{v}_{3b}) = E_1M_1\cosh(-\alpha_1h)/m_1+iF_1N_1\cosh(-\beta_1h)/n_1$$

$$+G_1\sinh(-\alpha_1h)+H_1\sinh(-\beta_1h)$$

$$-C_3\exp(-\alpha_3h)-iD_3\exp(-\beta_3h) \tag{31}$$

with

$$M_i = (1+c_{12})\alpha_i\xi, \quad N_i = (1+c_{12})\beta_i\xi. \tag{32}$$

Applying eqns (25), (30) and (31), coefficients C_2, D_2, C_3, D_3 are shown by

$$\begin{vmatrix} C_2/m_2 \\ D_2/n_2 \\ C_3/m_3 \\ D_3/n_3 \end{vmatrix} = \begin{vmatrix} H_{11} & H_{21} & H_{31} & H_{41} \\ H_{12} & H_{22} & H_{32} & H_{42} \\ H_{13} & H_{23} & H_{33} & H_{43} \\ H_{14} & H_{24} & H_{34} & H_{44} \end{vmatrix} \begin{vmatrix} \sum_{n=1}^{\infty} a_n(2n/\xi)J_{2n}(a\xi) \\ \sum_{n=1}^{\infty} b_n\{(2n-1)/\xi\}J_{(2n-1)}(a\xi) \\ \sum_{n=1}^{\infty} c_n(2n/\xi)J_{2n}(b\xi) \\ \sum_{n=1}^{\infty} d_n\{(2n-1)/\xi\}J_{(2n-1)}(b\xi) \end{vmatrix} \tag{33}$$

where H_{ij} is the cofactor of the element h_{ij} of the determinant Δ_1

$$\Delta_1 = \begin{vmatrix} h_{11} & h_{12} & \dots & h_{14} \\ h_{21} & & & \vdots \\ \vdots & & & \vdots \\ h_{41} & \dots & \dots & h_{44} \end{vmatrix} \tag{34}$$

with

$$\begin{aligned}
h_{11} &= M_1 f_9(\xi) + N_1 f_{13}(\xi) - M_2, \quad h_{12} = M_1 f_{10}(\xi) + N_1 f_{14}(\xi) - N_2 \\
h_{13} &= M_1 f_{11}(\xi) + N_1 f_{15}(\xi), \quad h_{14} = M_1 f_{12}(\xi) + N_1 f_{16}(\xi) \\
h_{21} &= m_1 f_1(\xi) + n_1 f_5(\xi) - m_2, \quad h_{22} = m_1 f_2(\xi) + n_1 f_6(\xi) - n_2 \\
h_{23} &= m_1 f_3(\xi) + n_1 f_7(\xi), \quad h_{24} = m_1 f_4(\xi) + n_1 f_8(\xi) \\
h_{31} &= M_1 f_9(\xi)\cosh(-\alpha_1 h) + N_1 f_{13}(\xi)\cosh(-\beta_1 h) \\
&\quad + M_1 f_1(\xi)\sinh(-\alpha_1 h) + N_1 f_5(\xi)\sinh(-\beta_1 h) \\
h_{32} &= M_1 f_{10}(\xi)\cosh(-\alpha_1 h) + N_1 f_{14}(\xi)\cosh(-\beta_1 h) \\
&\quad + M_1 f_2(\xi)\sinh(-\alpha_1 h) + N_1 f_6(\xi)\sinh(-\beta_1 h) \\
h_{33} &= M_1 f_{11}(\xi)\cosh(-\alpha_1 h) + N_1 f_{15}(\xi)\cosh(-\beta_1 h) \\
&\quad + M_1 f_3(\xi)\sinh(-\alpha_1 h) + N_1 f_7(\xi)\sinh(-\beta_1 h) - M_1 \exp(-\alpha_1 h) \\
h_{34} &= M_1 f_{12}(\xi)\cosh(-\alpha_1 h) + N_1 f_{16}(\xi)\cosh(-\beta_1 h) \\
&\quad + M_1 f_4(\xi)\sinh(-\alpha_1 h) + N_1 f_8(\xi)\sinh(-\beta_1 h) - M_1 \exp(-\beta_1 h) \\
h_{41} &= m_1 f_1(\xi)\cosh(-\alpha_1 h) + n_1 f_5(\xi)\cosh(-\beta_1 h) \\
&\quad + m_1 f_9(\xi)\sinh(-\alpha_1 h) + n_1 f_{13}(\xi)\sinh(-\beta_1 h) \\
h_{42} &= m_1 f_2(\xi)\cosh(-\alpha_1 h) + n_1 f_6(\xi)\cosh(-\beta_1 h) \\
&\quad + m_1 f_{10}(\xi)\sinh(-\alpha_1 h) + n_1 f_{14}(\xi)\sinh(-\beta_1 h) \\
h_{43} &= m_1 f_3(\xi)\cosh(-\alpha_1 h) + n_1 f_7(\xi)\cosh(-\beta_1 h) \\
&\quad + m_1 f_{11}(\xi)\sinh(-\alpha_1 h) + n_1 f_{15}(\xi)\sinh(-\beta_1 h) - m_1 \exp(-\alpha_1 h) \\
h_{44} &= m_1 f_1(\xi)\cosh(-\alpha_1 h) + n_1 f_8(\xi)\cosh(-\beta_1 h) \\
&\quad + m_1 f_{12}(\xi)\sinh(-\alpha_1 h) + n_1 f_{16}(\xi)\sinh(-\beta_1 h) - n_1 \exp(-\beta_1 h).
\end{aligned} \tag{35}$$

Substituting eqn (33) into the stress expressions and applying eqn (11b), we obtain

$$\pi\sigma_{y2a} = \sum_{n=1}^{\infty} a_n 2n \int_0^{\infty} \{Q_1(\xi)/\xi\} J_{2n}(a\xi)\cos(\xi x)\,\mathrm{d}\xi$$

$$+ \sum_{n=1}^{\infty} b_n(2n-1) \int_0^{\infty} \{Q_2(\xi)/\xi\} J_{2n-1}(a\xi)\cos(\xi x)\,\mathrm{d}\xi$$

$$+ \sum_{n=1}^{\infty} c_n 2n \int_0^{\infty} \{Q_3(\xi)/\xi\} J_{2n}(b\xi)\cos(\xi x)\,\mathrm{d}\xi$$

$$+ \sum_{n=1}^{\infty} d_n(2n-1) \int_0^{\infty} \{Q_4(\xi)/\xi\} J_{2n-1}(b\xi)\cos(\xi x)\,\mathrm{d}\xi$$

$$\pi\tau_{xy2a} = \sum_{n=1}^{\infty} a_n 2n \int_0^{\infty} \{Q_5(\xi)/\xi\} J_{2n}(a\xi)\sin(\xi x)\,\mathrm{d}\xi$$

$$+ \sum_{n=1}^{\infty} b_n(2n-1) \int_0^{\infty} \{Q_6(\xi)/\xi\} J_{2n-1}(a\xi)\sin(\xi x)\,\mathrm{d}\xi$$

$$+ \sum_{n=1}^{\infty} c_n 2n \int_0^{\infty} \{Q_7(\xi)/\xi\} J_{2n}(b\xi)\sin(\xi x)\,\mathrm{d}\xi$$

$$+ \sum_{n=1}^{\infty} d_n(2n-1) \int_0^{\infty} \{Q_8(\xi)/\xi\} J_{2n-1}(b\xi)\sin(\xi x)\,\mathrm{d}\xi$$

$$\pi\sigma_{y3b} = \sum_{n=1}^{\infty} a_n 2n \int_0^{\infty} \{Q_9(\xi)/\xi\} J_{2n}(a\xi)\cos(\xi x)\,\mathrm{d}\xi$$

$$+ \sum_{n=1}^{\infty} b_n(2n-1) \int_0^{\infty} \{Q_{10}(\xi)/\xi\} J_{2n-1}(a\xi)\cos(\xi x)\,\mathrm{d}\xi$$

$$+ \sum_{n=1}^{\infty} c_n 2n \int_0^{\infty} \{Q_{11}(\xi)/\xi\} J_{2n}(b\xi)\cos(\xi x)\,\mathrm{d}\xi$$

$$+ \sum_{n=1}^{\infty} d_n(2n-1) \int_0^{\infty} \{Q_{12}(\xi)/\xi\} J_{2n-1}(b\xi)\cos(\xi x)\,\mathrm{d}\xi$$

$$\pi\tau_{xy3b} = \sum_{n=1}^{\infty} a_n 2n \int_0^{\infty} \{Q_{13}(\xi)/\xi\} J_{2n}(a\xi)\sin(\xi x)\,\mathrm{d}\xi$$

$$+ \sum_{n=1}^{\infty} b_n(2n-1) \int_0^{\infty} \{Q_{14}(\xi)/\xi\} J_{2n-1}(a\xi)\sin(\xi x)\,\mathrm{d}\xi$$

$$+ \sum_{n=1}^{\infty} c_n 2n \int_0^{\infty} \{Q_{15}(\xi)/\xi\} J_{2n}(b\xi)\sin(\xi x)\,\mathrm{d}\xi$$

$$+ \sum_{n=1}^{\infty} d_n(2n-1) \int_0^{\infty} \{Q_{16}(\xi)/\xi\} J_{2n-1}(b\xi)\sin(\xi x)\,\mathrm{d}\xi \qquad (36)$$

where $Q_1(\xi)$, $Q_2(\xi)$, ..., $Q_{16}(\xi)$ are given in Appendix B.

Now, the remaining boundary conditions are eqns (9a) and (10a), and these are reduced to the forms

$$\sum_{n=1}^{\infty} a_n K_{na}(x) + \sum_{n=1}^{\infty} b_n L_{na}(x) + \sum_{n=1}^{\infty} c_n M_{na}(x) + \sum_{n=1}^{\infty} d_n N_{na}(x) = -U(x)$$

$$\sum_{n=1}^{\infty} a_n O_{na}(x) + \sum_{n=1}^{\infty} b_n P_{na}(x) + \sum_{n=1}^{\infty} c_n Q_{na}(x) + \sum_{n=1}^{\infty} d_n R_{na}(x) = -V(x) \quad (0 \leqq x < a) \quad (37a)$$

$$\sum_{n=1}^{\infty} a_n K_{nb}(x) + \sum_{n=1}^{\infty} b_n L_{nb}(x) + \sum_{n=1}^{\infty} c_n M_{nb}(x) + \sum_{n=1}^{\infty} d_n N_{nb}(x) = -W(x)$$

$$\sum_{n=1}^{\infty} a_n O_{nb}(x) + \sum_{n=1}^{\infty} b_n P_{nb}(x) + \sum_{n=1}^{\infty} c_n Q_{nb}(x) + \sum_{n=1}^{\infty} d_n R_{nb}(x) = -Z(x) \quad (0 \leqq x < b) \quad (37b)$$

with

$$U(x) = p_0(a-x)^{1/2}, \quad V(x) = 0$$
$$W(x) = p_0(b-x)^{1/2} \exp\{-i\omega h/(c_1\sqrt{c_{22}})\}, \quad Z(x) = 0 \quad (38)$$

and $K_{na}(x)$, $L_{na}(x)$, ..., $R_{nb}(x)$ are shown in Appendix C. Equation (37) can now be solved for coefficients a_n, b_n, c_n, d_n using the Schmidt method (Itou, 1976) as described in Appendix D.

5. STRESS INTENSITY FACTORS

Stresses at $y = 0$ and at $y = -h$ are given by eqn (36). The singularities in the stress field result from the relationships

$$\int_0^{\infty} J_n(a\xi)[\cos(\xi x), \sin(\xi x)]\, d\xi = [-a^n(x^2-a^2)^{-1/2}\{x+(x^2-a^2)^{1/2}\}^{-n} \sin(n\pi/2),$$
$$a^n(x^2-a^2)^{-1/2}\{x+(x^2-a^2)^{1/2}\}^{-n} \cos(n\pi/2)] \quad \textit{for } a < x. \quad (39)$$

Then, we can easily define stress intensity factors K_{1a}, K_{2a}, K_{1b}, K_{2b} as follows

$$K_{1a} = \lim_{x \to a+} \{2\pi(x-a)\}^{1/2} \sigma_{y2a}$$
$$= \sum_{n=1}^{\infty} b_n(2n-1)(-1)^n Q_2^L/(\pi a)^{1/2}$$
$$K_{2a} = \lim_{x \to a+} \{2\pi(x-a)\}^{1/2} \tau_{xy2a}$$
$$= \sum_{n=1}^{\infty} a_n 2n(-1)^n Q_5^L/(\pi a)^{1/2}$$
$$K_{1b} = \lim_{x \to b+} \{2\pi(x-b)\}^{1/2} \sigma_{y3b}$$
$$= \sum_{n=1}^{\infty} d_n(2n-1)(-1)^n Q_{12}^L/(\pi b)^{1/2}$$
$$K_{2b} = \lim_{x \to b+} \{2\pi(x-b)\}^{1/2} \tau_{xy3b}$$
$$= \sum_{n=1}^{\infty} c_n 2n(-1)^n Q_{15}^L/(\pi b)^{1/2} \quad (40)$$

where Q_2^L, Q_5^L, Q_{12}^L, Q_{15}^L are given by

$$Q_i^L = Q_i(\xi_L)/\xi_L \tag{41}$$

with ξ_L being a large value of ξ. Expressions Q_i^L cannot be expressed in terms of a formula and must thus be obtain numerically, as seen in the next section.

6. NUMERICAL EXAMPLES AND RESULTS

Numerical calculations are carried out for a boron–epoxy composite, a carbon fiber reinforced plastic, a modulite II graphite–epoxy composite and an isotropic material. The material constants are shown in Table 1. In the table, the fourth set of elastic constants are not those for the specific material. These are only used to obtain the numerical results for an isotropic material. The infinite integrals in functions $k_{na}(x)$, $l_{na}(x)$, $\ldots$, $q_{nb}(x)$, $r_{nb}(x)$ in eqn (C.3) are calculated taking the upper limit of integration up to (ξa) = 10.0, 12.0, 13.0 and 40.0 for the boron–epoxy composite, carbon fiber reinforced plastic, modulite II graphite–epoxy composite and isotropic material, respectively.

In Table 2, the values of λ_1 and λ_3 are given for $\omega a/c_t$ = 1.1. From the table, the critical values of ξ are known at which the roots of eqn (14) change types. The values of

Table 1. Material constants

Material	E_x (N/m²)	E_y (N/m²)	μ_{xy} (N/m²)	ν_{xy}
Boron–epoxy composite	224.06×10^9	12.69×10^9	4.43×10^9	0.256
Carbon fiber reinforced plastic	145.0×10^9	9.6×10^9	4.8×10^9	0.23
Modulite II graphite–epoxy composite	158.0×10^9	15.3×10^9	5.52×10^9	0.34
Isotropic material	100.0×10^9	100.0×10^9	37.5×10^9	1/3

Table 2. Values of λ_1 and λ_3 for $\omega a/c_t$ = 1.1

Material	(ξa)	$(\lambda_1 a)$	$(\lambda_3 a)$
Boron–epoxy composite	0.12	0.0 +0.61480 i	0.0 +0.75320 i
	0.14	0.0 +0.47968 i	0.0 +0.67943 i
	0.16	0.20298+0.0 i	0.0 +0.66528 i
	1.08	7.48993+0.0 i	0.0 +0.13178 i
	1.10	7.63188+0.0 i	0.0 +0.04231 i
	1.12	7.77376+0.0 i	0.11863+0.0 i
Carbon fiber reinforced plastic	0.18	0.0 +0.48650 i	0.0 +0.80466 i
	0.20	0.0 +0.17558 i	0.0 +0.79469 i
	0.22	0.45152+0.0 i	0.0 +0.78792 i
	1.08	5.70220+0.0 i	0.0 +0.15895 i
	1.10	5.81185+0.0 i	0.0 +0.05103 i
	1.12	5.92413+0.0 i	0.14308+0.0 i
Modulite II graphite-epoxy composite	0.18	0.0 +0.50395 i	0.0 +0.73146 i
	0.20	0.0 +0.29901 i	0.0 +0.70324 i
	0.22	0.34299+0.0 i	0.0 +0.68900 i
	1.08	5.48477+0.0 i	0.0 +0.13649 i
	1.10	5.59053+0.0 i	0.0 +0.04382 i
	1.12	5.69622+0.0 i	0.12286+0.0 i
Isotropic material (Ey/Ex = 1.0, ν_{xy} = 1/3)	0.68	0.0 +0.19391 i	0.0 +1.01059 i
	0.70	0.0 +0.13063 i	0.0 +1.00111 i
	0.72	0.05742+0.0 i	0.0 +0.99086 i
	1.08	0.39436+0.0 i	0.0 +0.45062 i
	1.10	0.25880+0.0 i	0.0 +0.22771 i
	1.12	0.32395+0.25587 i	0.32395−0.25587 i
	1.14	0.40571+0.30437 i	0.40571−0.30437 i

Table 3. Values of $Q_i(\xi a)/(\xi a)$ for $\omega a/c_t = 1.1$, $h/a = 1.0$, $b/a = 1.0$

Material	(ξa)	$Q_2(\xi a)/(\xi a)$	$Q_5(\xi a)/(\xi a)$	$Q_{12}(\xi a)/(\xi a)$	$Q_{15}(\xi a)/(\xi a)$
Boron–epoxy	9.92	0.34487×10^1	0.14579×10^2	-0.34487×10^1	-0.14579×10^2
composite	9.94	0.34488×10^1	0.14579×10^2	-0.34488×10^1	-0.14579×10^2
	9.96	0.34489×10^1	0.14579×10^2	-0.34489×10^1	-0.14579×10^2
	9.98	0.34490×10^1	0.14579×10^2	-0.34490×10^1	-0.14579×10^2
	10.00	0.34491×10^1	0.14580×10^2	-0.34491×10^1	-0.14580×10^2
Carbon fiber	11.92	0.30179×10^1	0.11779×10^2	-0.30179×10^1	-0.11779×10^2
reinforced	11.94	0.30179×10^1	0.11779×10^2	-0.30179×10^1	-0.11779×10^2
plastic	11.96	0.30180×10^1	0.11779×10^2	-0.30180×10^1	-0.11779×10^2
	11.98	0.30180×10^1	0.11779×10^2	-0.30180×10^1	-0.11779×10^2
	12.00	0.30181×10^1	0.11779×10^2	-0.30181×10^1	-0.11779×10^2
Modulite II	12.92	0.41215×10^1	0.13291×10^2	-0.41215×10^1	-0.13291×10^2
graphite–epoxy	12.94	0.41216×10^1	0.13291×10^2	-0.41216×10^1	-0.13291×10^2
composite	12.96	0.41216×10^1	0.13291×10^2	-0.41216×10^1	-0.13291×10^2
	12.98	0.41217×10^1	0.13291×10^2	-0.41217×10^1	-0.13291×10^2
	13.00	0.41217×10^1	0.13291×10^2	-0.41217×10^1	-0.13291×10^2
Isotropic	39.92	0.21287×10^2	0.21313×10^2	-0.21287×10^2	-0.21313×10^2
material	39.94	0.21287×10^2	0.21313×10^2	-0.21287×10^2	-0.21313×10^2
($Ey/Ex = 1.0$, $\nu_{xy} = 1/3$)	39.96	0.21287×10^2	0.21313×10^2	-0.21287×10^2	-0.21313×10^2
	39.98	0.21287×10^2	0.21313×10^2	-0.21287×10^2	-0.21313×10^2
	40.00	0.21287×10^2	0.21313×10^2	-0.21287×10^2	-0.21313×10^2

$Q_i^L\{=Q_i(\xi a)/(\xi a)\}$ are shown in Table 3 for $\omega a/c_t = 1.1$, $h/a = 1.0$, $b/a = 1.0$. From the table, it is obvious that the values of $Q_i(\xi a)/(\xi a)$ asymptotically approach the constants for $i = 2, 5, 12, 15$. Values of $Q_i(\xi a)/(\xi a)$ for $i = 3, 4, 7, 8, 9, 10, 13, 14$ are not shown in the tables. However, it has been verified that these values decay rapidly when (ξa) increases. Thus, the semi-infinite integrals in eqn (C.3) can be easily evaluated using Filon's method (Amemiya and Taguchi, 1969).

The infinite series in eqn (37) can be truncated by summing from $n = 1$ to 7. In Table 4, the values for lhs and those for rhs in eqn (37a) are shown for the boron–epoxy composite with $\omega a/c_t = 1.1$, $h/a = 1.0$, $b/a = 1.0$. Table 5 shows the values in eqn (37b) for the same case. From the tables, it can be seen that the Schmidt method has been applied satisfactorily.

Table 4. Values of lhs and rhs in eqn (37a) for boron–epoxy composite ($\omega a/c_t = 1.1$, $h/a = 1.0$, $b/a = 1.0$)

x/a	$\sum_{n=1}^{7}\{a_nK_{na}(x/a)+b_nL_{na}(x/a)$ $+c_nM_{na}(x/a)+d_nN_{na}(x/a)\}/(p_o\sqrt{a})$	$\sum_{n=1}^{7}\{a_nO_{na}(x/a)+b_nP_{na}(x/a)$ $+c_nQ_{na}(x/a)+d_nR_{na}(x/a)\}/(p_o\sqrt{a})$	$-U(x/a)/(p_o\sqrt{a})$
0.00010	$-0.99911+0.00001\,i$	$0.00000+0.00000\,i$	$-0.99995+0.00000\,i$
0.07143	$-0.96361+0.00000\,i$	$0.00000+0.00000\,i$	$-0.96362+0.00000\,i$
0.50000	$-0.70713+0.00001\,i$	$-0.00000+0.00000\,i$	$-0.70711+0.00000\,i$
0.92857	$-0.26726+0.00000\,i$	$-0.00000+0.00000\,i$	$-0.26720+0.00000\,i$
0.99990	$-0.01009+0.00003\,i$	$-0.00001-0.00000\,i$	$-0.01000+0.00000\,i$

Table 5. Values of lhs and rhs in eqn (37b) for boron–epoxy composite ($\omega a/c_t = 1.1$, $h/a = 1.0$, $b/a = 1.0$)

x/a	$\sum_{n=1}^{7}\{a_nK_{nb}(x/a)+b_nL_{nb}(x/a)$ $+c_nM_{nb}(x/a)+d_nN_{nb}(x/a)\}/(p_o\sqrt{a})$	$\sum_{n=1}^{7}\{a_nO_{nb}(x/a)+b_nP_{nb}(x/a)$ $+c_nQ_{nb}(x/a)+d_nR_{nb}(x/a)\}/(p_o\sqrt{a})$	$-W(x/a)/(p_o\sqrt{a})$
0.00010	$-0.79538+0.60604\,i$	$0.00000+0.00000\,i$	$-0.79536+0.60606\,i$
0.07143	$-0.76647+0.58404\,i$	$0.00000+0.00000\,i$	$-0.76646+0.58404\,i$
0.50000	$-0.56240+0.42859\,i$	$-0.00000-0.00000\,i$	$-0.56243+0.42857\,i$
0.92857	$-0.21258+0.16198\,i$	$0.00000+0.00000\,i$	$-0.21258+0.16198\,i$
0.99990	$-0.00786+0.00612\,i$	$-0.00001-0.00000\,i$	$-0.00795+0.00606\,i$

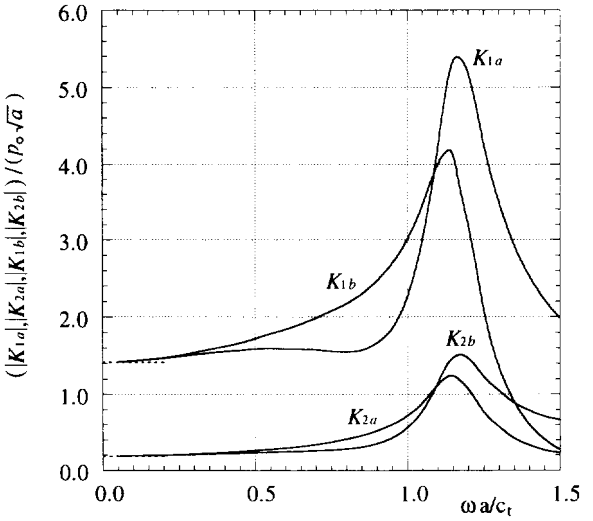

Fig. 2. Absolute values of K_{1a}, K_{2a}, K_{1b}, K_{2b} for boron–epoxy material (b/a = 1.0 and h/a = 1.0).

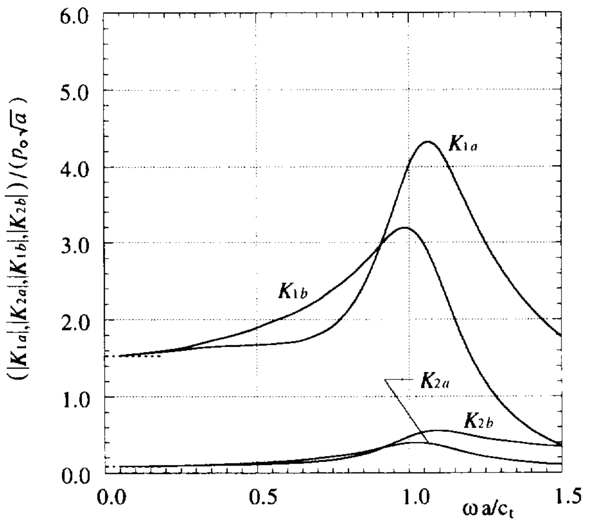

Fig. 3. Absolute values of K_{1a}, K_{2a}, K_{1b}, K_{2b} for boron–epoxy composite material (b/a = 1.0 and h/a = 2.0) .

The absolute values of the dynamic stress intensity factors are plotted in Figs 2–9, where the broken straight lines indicate the corresponding static values. Numerical results for the isotropic material with a Poisson's ratio of $\nu_{xy} = 1/3$ in the plane stress condition correspond to those for $\nu = 0.25$ in the plane strain condition. The curves for the isotropic material with $h/a = 1.0$ and $\nu_{xy} = 1/3$ are in good agreement with the results for $\nu = 0.25$ calculated in the plane strain condition by Takakuda (1982) using the integral equation method. The absolute values of the $K_{1a}^{peak}/K_{1a}^{static}$ ratio and K_{1a}^{peak} are shown in Table 6.

It is apparent from the figures that the predominant term of the dynamic stress intensity factor is K_{1a}. As can be seen in Table 6, the K_{1a}^{peak} values are considerably larger than the K_{1a}^{static} values.

For a value of h/a smaller than 1.0, more terms are needed in order to properly apply the Schmidt method. In such cases, an overflow occurs in the numerical computations, and numerical results are therefore not obtainable. However, it can be inferred from Table 6 that the $K_{1a}^{peak}/K_{1a}^{static}$ ratio increases extremely as h/a decreases.

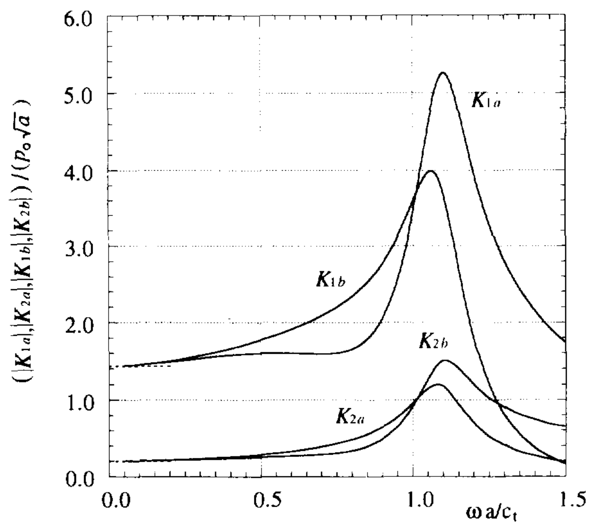

Fig. 4. Absolute values of K_{1a}, K_{2a}, K_{1b}, K_{2b} for carbon fiber reinforced plastic ($b/a = 1.0$ and $h/a = 1.0$).

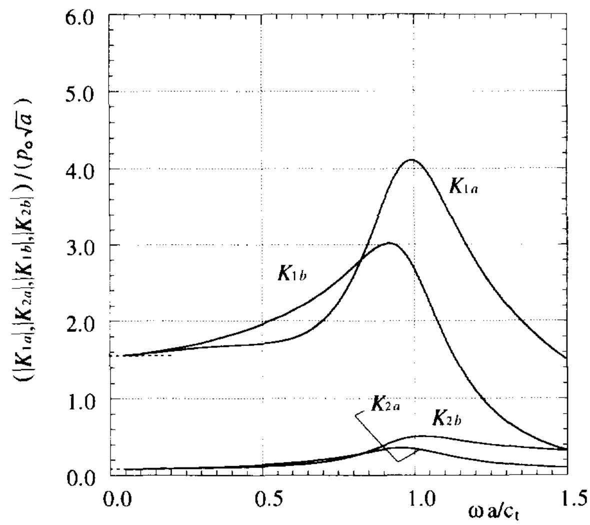

Fig. 5. Absolute values of K_{1a}, K_{2a}, K_{1b}, K_{2b} for carbon fiber reinforced plastic ($b/a = 1.0$ and $h/a = 2.0$).

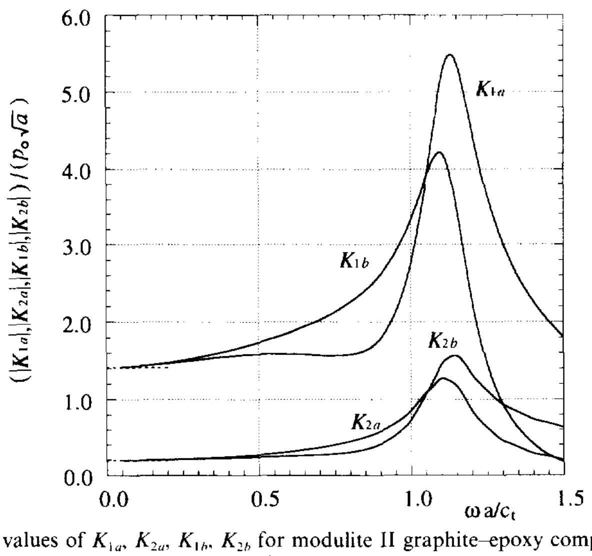

Fig. 6. Absolute values of K_{1a}, K_{2a}, K_{1b}, K_{2b} for modulite II graphite–epoxy composite ($b/a = 1.0$ and $h/a = 1.0$).

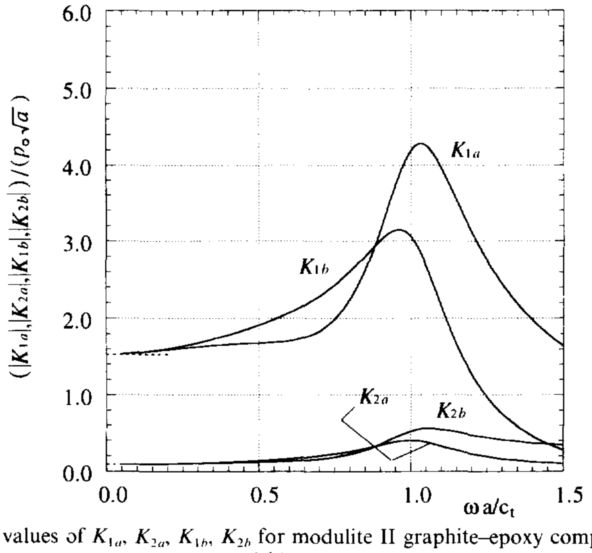

Fig. 7. Absolute values of K_{1a}, K_{2a}, K_{1b}, K_{2b} for modulite II graphite–epoxy composite ($b/a = 1.0$ and $h/a = 2.0$).

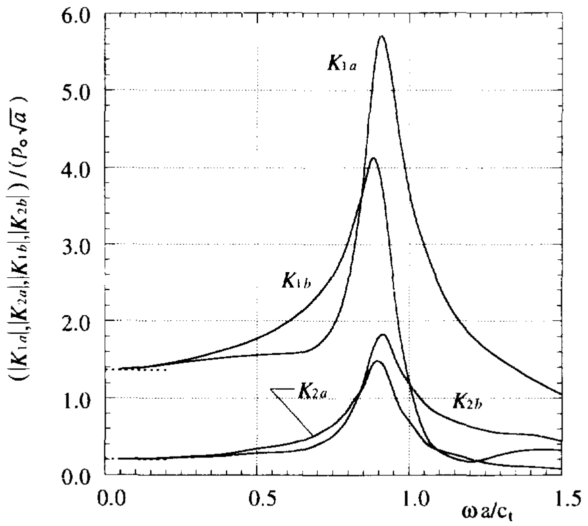

Fig. 8. Absolute values of K_{1a}, K_{2a}, K_{1b}, K_{2b} for isotropic material ($b/a = 1.0$ and $h/a = 1.0$).

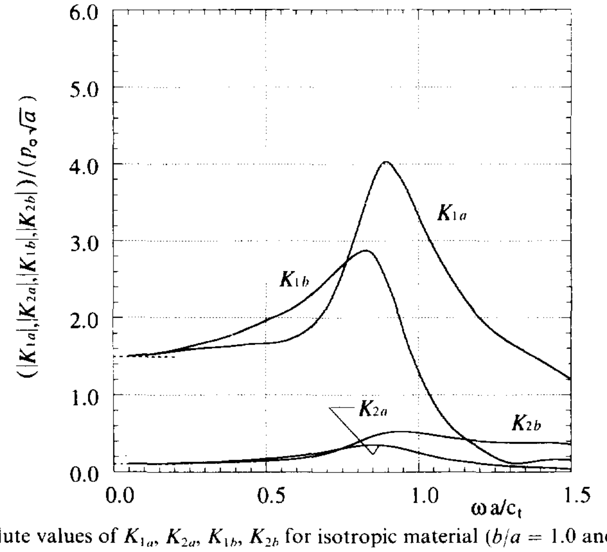

Fig. 9. Absolute values of K_{1a}, K_{2a}, K_{1b}, K_{2b} for isotropic material ($b/a = 1.0$ and $h/a = 2.0$).

Table 6. Absolute values of $K_{1a}^{peak}/K_{1a}^{static}$ and K_{1a}^{peak} for $a/b = 1.0$.

	Boron–epoxy composite	Carbon fiber reinforced plastic	Modulite II graphite–epoxy composite	Isotropic material
$\lvert K_{1a}^{peak}\rvert/K_{1a}^{static}$ $h/a = 1.0$	3.81	3.67	3.87	4.16
$\lvert K_{1a}^{peak}\rvert/K_{1a}^{static}$ $h/a = 2.0$	2.82	2.64	2.78	2.70
$\lvert K_{1a}^{peak}\rvert/(p_o\sqrt{a})$ $h/a = 1.0$	5.40	5.25	5.47	5.70
$\lvert K_{1a}^{peak}\rvert/(p_o\sqrt{a})$ $h/a = 2.0$	4.31	4.11	4.26	4.03

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APPENDIX A

$$f_{4(i-1)+j}(\xi) = \begin{vmatrix} a_{11}\dots a_{1i-1} & b_{1j} & a_{1i+1}\dots a_{14} \\ a_{21}\dots a_{2i-1} & b_{2j} & a_{2i+1}\dots a_{24} \\ a_{31}\dots a_{3i-1} & b_{3j} & a_{3i+1}\dots a_{34} \\ a_{41}\dots a_{4i-1} & b_{4j} & a_{4i+1}\dots a_{44} \end{vmatrix} \Big/ \Delta_2$$

$$\text{for } i = 1,2,3,4 \text{ and } j = 1,2,3,4 \tag{A.1}$$

with

$$\Delta_2 = |a_{kl}| \tag{A.2}$$

$$a_{11} = 0, \quad a_{12} = 0, \quad a_{13} = c_{12}(1+c_{12})\alpha_1\xi^2 + c_{22}m_1\alpha_1$$

$$a_{14} = c_{12}(1+c_{12})\beta_1\xi^2 + c_{22}n_1\beta_1, \quad b_{11} = c_{12}(1+c_{12})\alpha_2\xi^2 + c_{22}m_2\alpha_2$$

$$b_{12} = c_{12}(1+c_{12})\beta_2\xi^2 + c_{22}n_2\beta_2, \quad b_{13} = 0, \quad b_{14} = 0$$

$$a_{21} = (1+c_{12})\alpha_1^2\xi - m_1\xi, \quad a_{22} = (1+c_{12})\beta_1^2\xi - n_1\xi, \quad a_{23} = 0, \quad a_{24} = 0$$

$$b_{21} = (1+c_{12})\alpha_2^2\xi - m_1\xi, \quad b_{22} = (1+c_{12})\beta_2^2\xi - n_2\xi, \quad b_{23} = 0, \quad b_{24} = 0$$

$$a_{31} = \{c_{12}(1+c_{12})\alpha_1\xi^2 + c_{22}m_1\alpha_1\}\sinh(-\alpha_1 h)$$

$$a_{32} = \{c_{12}(1+c_{12})\beta_1\xi^2 + c_{22}n_1\beta_1\}\sinh(-\beta_1 h)$$

$$a_{33} = \{c_{12}(1+c_{12})\alpha_1\xi^2 + c_{22}m_1\alpha_1\}\cosh(-\alpha_1 h)$$

$$a_{34} = \{c_{12}(1+c_{12})\beta_1\xi^2 + c_{22}n_1\beta_1\}\cosh(-\beta_1 h), \quad b_{31} = 0, \quad b_{32} = 0$$

$$b_{33} = \{c_{12}(1+c_{12})\alpha_1\xi^2 + c_{22}m_1\alpha_1\}\exp(-\alpha_1 h)$$

$$b_{34} = \{c_{12}(1+c_{12})\beta_1\xi^2 + c_{22}n_1\beta_1\}\exp(-\beta_1 h)$$

$$
\begin{aligned}
a_{41} &= \{(1+c_{12})\alpha_1^2\xi - m_1\xi\}\cosh(-\alpha_1 h) \\
a_{42} &= \{(1+c_{12})\beta_1^2\xi - n_1\xi\}\cosh(-\beta_1 h) \\
a_{43} &= \{(1+c_{12})\alpha_1^2\xi - m_1\xi\}\sinh(-\alpha_1 h) \\
a_{44} &= \{(1+c_{12})\beta_1^2\xi - n_1\xi\}\sinh(-\beta_1 h), \quad b_{41} = 0, \quad b_{42} = 0 \\
b_{43} &= \{(1+c_{12})\alpha_1^2\xi - m_1\xi\}\exp(-\alpha_1 h) \\
b_{44} &= \{(1+c_{12})\beta_1^2\xi - n_1\xi\}\exp(-\beta_1 h).
\end{aligned} \tag{A3}
$$

APPENDIX B

$$
\begin{aligned}
&Q_1(\xi) = H_{11}s_1 + H_{12}s_2, \quad Q_2(\xi) = H_{21}s_1 + H_{22}s_2, \quad Q_3(\xi) = H_{31}s_1 + H_{32}s_2 \\
&Q_4(\xi) = H_{41}s_1 + H_{42}s_2, \quad Q_5(\xi) = H_{11}s_3 + H_{12}s_4, \quad Q_6(\xi) = H_{21}s_3 + H_{22}s_4 \\
&Q_7(\xi) = H_{31}s_3 + H_{32}s_4, \quad Q_8(\xi) = H_{41}s_3 + H_{42}s_4, \quad Q_9(\xi) = H_{13}s_5 + H_{14}s_6 \\
&Q_{10}(\xi) = H_{23}s_5 + H_{24}s_6, \quad Q_{11}(\xi) = H_{33}s_5 + H_{34}s_6, \quad Q_{12}(\xi) = H_{43}s_5 + H_{44}s_6 \\
&Q_{13}(\xi) = H_{13}s_7 + H_{14}s_8, \quad Q_{14}(\xi) = H_{23}s_7 + H_{24}s_8, \quad Q_{15}(\xi) = H_{33}s_7 + H_{34}s_8 \\
&Q_{16}(\xi) = H_{43} + H_{44}s_8
\end{aligned} \tag{B.1}
$$

with

$$
\begin{aligned}
s_1 &= \mu_{xy}(c_{12}\xi M_2 + c_{22}m_2\alpha_2), \quad s_2 = \mu_{xy}(c_{12}\xi N_2 + c_{22}n_2\beta_2) \\
s_3 &= \mu_{xy}(M_2\alpha_2 - m_2\xi), \quad s_4 = \mu_{xy}(N_2\beta_2 - n_2\xi) \\
s_5 &= \mu_{xy}(c_{12}\xi M_3 + c_{22}m_3\alpha_3)\exp(-\alpha_3 h) \\
s_6 &= \mu_{xy}(c_{12}\xi N_3 + c_{22}n_3\beta_3)\exp(-\beta_3 h) \\
s_7 &= \mu_{xy}(M_3\alpha_3 - m_3\xi)\exp(-\alpha_3 h), \quad s_8 = \mu_{xy}(N_3\beta_3 - n_3\xi)\exp(-\beta_3 h).
\end{aligned} \tag{B.2}
$$

APPENDIX C

$$
\begin{aligned}
K_{na}(x) &= (x-a)^{1/2}k_{na}(x), \quad L_{na}(x) = (x-a)^{1/2}l_{na}(x) \\
M_{na}(x) &= (x-a)^{1/2}m_{na}(x), \quad N_{na}(x) = (x-a)^{1/2}n_{na}(x) \\
O_{na}(x) &= (x-a)^{1/2}o_{na}(x), \quad P_{na}(x) = (x-a)^{1/2}p_{na}(x) \\
Q_{na}(x) &= (x-a)^{1/2}q_{na}(x), \quad R_{na}(x) = (x-a)^{1/2}r_{na}(x)
\end{aligned} \tag{C.1}
$$

$$
\begin{aligned}
K_{nb}(x) &= (x-b)^{1/2}k_{nb}(x), \quad L_{nb}(x) = (x-b)^{1/2}l_{nb}(x) \\
M_{nb}(x) &= (x-b)^{1/2}m_{nb}(x), \quad N_{nb}(x) = (x-b)^{1/2}n_{nb}(x) \\
O_{nb}(x) &= (x-b)^{1/2}o_{nb}(x), \quad P_{nb}(x) = (x-b)^{1/2}p_{nb}(x) \\
Q_{nb}(x) &= (x-b)^{1/2}q_{nb}(x), \quad R_{nb}(x) = (x-b)^{1/2}r_{nb}(x)
\end{aligned} \tag{C.2}
$$

with

$$
\begin{aligned}
k_{na}(x) &= 0 \\
l_{na}(x) &= \{(2n-1)/\pi\}\left[\int_0^\infty \{Q_2(\xi)/\xi - Q_2^L\}J_{2n-1}(a\xi)\cos(\xi x)\,d\xi \right. \\
&\qquad \left. + Q_2^L(a^2-x^2)^{-1/2}\cos\{(2n-1)\sin^{-1}(x/a)\}\right] \\
m_{na}(x) &= (2n/\pi)\int_0^\infty \{Q_3(\xi)/\xi\}J_{2n}(b\xi)\cos(\xi x)\,d\xi \\
n_{na}(x) &= \{(2n-1)/\pi\}\int_0^\infty \{Q_4(\xi)/\xi\}J_{2n-1}(b\xi)\cos(\xi x)\,d\xi
\end{aligned}
$$

$$o_{na}(x) = (2n/\pi)\left[\int_0^\infty \{Q_5(\xi)/\xi - Q_5^L\}J_{2n}(a\xi)\sin(\xi x)\,d\xi\right.$$

$$\left.+Q_5^L(a^2-x^2)^{-1/2}\sin\{2n\sin^{-1}(x/a)\}\right]$$

$$p_{na}(x) = 0$$

$$q_{na}(x) = (2n/\pi)\int_0^\infty \{Q_7(\xi)/\xi\}J_{2n}(b\xi)\sin(\xi x)\,d\xi$$

$$r_{na}(x) = \{(2n-1)/\pi\}\int_0^\infty \{Q_8(\xi)/\xi\}J_{2n-1}(b\xi)\sin(\xi x)\,d\xi$$

$$k_{nb}(x) = (2n/\pi)\int_0^\infty \{Q_9(\xi)/\xi\}J_{2n}(a\xi)\cos(\xi x)\,d\xi$$

$$l_{nb}(x) = \{(2n-1)/\pi\}\int_0^\infty \{Q_{10}(\xi)/\xi\}J_{2n-1}(a\xi)\cos(\xi x)\,d\xi$$

$$m_{nb} = 0$$

$$n_{nb}(x) = \{(2n-1)\pi\}\left[\int_0^\infty \{Q_{12}(\xi)/\xi - Q_{12}^L\}J_{2n-1}(b\xi)\cos(\xi x)\,d\xi\right.$$

$$\left.+Q_{12}^L(b^2-x^2)^{-1/2}\cos\{(2n-1)\sin^{-1}(x/b)\}\right]$$

$$o_{nb}(x) = (2n/\pi)\int_0^\infty \{Q_{13}(\xi)/\xi\}J_{2n}(a\xi)\sin(\xi x)\,d\xi$$

$$p_{nb}(x) = \{(2n-1)/\pi\}\int_0^\infty \{Q_{14}(\xi)/\xi\}J_{2n-1}(a\xi)\sin(\xi x)\,d\xi$$

$$q_{nb}(x) = (2n/\pi)\left[\int_0^\infty \{Q_{15}(\xi)/\xi - Q_{15}^L\}J_{2n}(b\xi)\sin(\xi x)\,d\xi\right.$$

$$\left.+Q_{15}^L(b^2-x^2)^{-1/2}\sin\{2n\sin^{-1}(x/b)\}\right]$$

$$r_{nb}(x) = 0 \tag{C3}$$

where Q_i^L are given by eqn (41).

APPENDIX D

For convenience, eqn (37) can be rewritten as

$$\sum_{n=1}^\infty a_nE_n(x)+\sum_{n=1}^\infty b_nF_n(x)+\sum_{n=1}^\infty c_nG_n(x)+\sum_{n=1}^\infty d_nH_n(x) = -U(x) \tag{D.1}$$

$$\sum_{n=1}^\infty a_nI_n(x)+\sum_{n=1}^\infty b_nJ_n(x)+\sum_{n=1}^\infty c_nK_n(x)+\sum_{n=1}^\infty d_nL_n(x) = -V(x) \quad \text{for } (x_1 \leqq x \leqq x_2) \tag{D.2}$$

$$\sum_{n=1}^\infty a_nM_n(y)+\sum_{n=1}^\infty b_nN_n(y)+\sum_{n=1}^\infty c_nO_n(y)+\sum_{n=1}^\infty d_nP_n(y) = -W(y) \tag{D.3}$$

$$\sum_{n=1}^\infty a_nQ_n(y)+\sum_{n=1}^\infty b_nR_n(y)+\sum_{n=1}^\infty c_nS_n(y)+\sum_{n=1}^\infty d_nT_n(y) = -Z(y) \quad \text{for } (y_1 \leqq y \leqq y_2) \tag{D.4}$$

where $E_n(x)$, $F_n(x)$, ..., $Z(y)$ are known functions and coefficients a_n, b_n, c_n and d_n are unknown and remain to be determined.

A set of functions $B_n(y)$ that satisfy the orthogonality condition

$$\int_{y_1}^{y_2} B_m(y)B_n(y)\,\mathrm{d}y = N_n\,\delta_{mn}, \quad N_n = \int_{y_1}^{y_2} B_n^2(y)\,\mathrm{d}y \tag{D.5}$$

can be constructed from a given set of arbitrary functions, say $R_n(y)$, such that

$$B_n(y) = \sum_{i=1}^{n} (M_{in}/M_{nn})R_i(y) \tag{D.6}$$

where M_{in} is the cofactor of the element d_{in} of D_n, which is defined as

$$D_n = \begin{vmatrix} d_{11} & d_{12} & \dots & d_{1n} \\ d_{21} & & & \vdots \\ \vdots & & & \vdots \\ d_{n1} & \dots & \dots & d_{nn} \end{vmatrix}, \quad d_{in} = \int_{y_1}^{y_2} R_i(y)R_n(y)\,\mathrm{d}y. \tag{D.7}$$

It is evident that N_n in eqn (D.5) is different from $N_n(y)$ in eqn (D.3). Representing the second series in eqn (D.4) by the orthogonal series $B_n(y)$ with coefficients e_n, the following relationships are derivable

$$\begin{aligned} \sum_{n=1}^{\infty} b_n R_n(y) &= \sum_{n=1}^{\infty} e_n B_n(y) \\ &= -Z(y) - \sum_{n=1}^{\infty} a_n Q_n(y) - \sum_{n=1}^{\infty} c_n S_n(y) - \sum_{n=1}^{\infty} d_n T_n(y). \end{aligned} \tag{D.8}$$

The second equality yields

$$\begin{aligned} e_n = -(1/N_n)\int_{y_1}^{y_2} \Big\{ & Z(y) + \sum_{i=1}^{\infty} a_i Q_i(y) \\ & + \sum_{i=1}^{\infty} c_i S_i(y) + \sum_{n=1}^{\infty} d_i T_i(y) \Big\} B_n(y)\,\mathrm{d}y \end{aligned} \tag{D.9}$$

and considering eqn (D.6), the first equality shows

$$b_n = \sum_{i=1}^{\infty} \gamma_{ni} a_i + \sum_{i=1}^{\infty} \delta_{ni}^{(1)} c_i + \sum_{n=1}^{\infty} \delta_{ni}^{(2)} d_i + \delta_n^{(0)} \tag{D.10}$$

with

$$\begin{aligned} \gamma_{ni} &= -\sum_{j=n}^{\infty} \{M_{nj}/(N_j M_{jj})\} \int_{y_1}^{y_2} Q_i(y) B_j(y)\,\mathrm{d}y \\ \delta_{ni}^{(1)} &= -\sum_{j=n}^{\infty} \{M_{nj}/(N_j M_{jj})\} \int_{y_1}^{y_2} S_i(y) B_j(y)\,\mathrm{d}y \\ \delta_{ni}^{(2)} &= -\sum_{j=n}^{\infty} \{M_{nj}/(N_j M_{jj})\} \int_{y_1}^{y_2} T_i(y) B_j(y)\,\mathrm{d}y \\ \delta_n^{(0)} &= -\sum_{j=n}^{\infty} \{M_{nj}/(N_j M_{jj})\} \int_{y_1}^{y_2} Z(y) B_j(y)\,\mathrm{d}y. \end{aligned} \tag{D.11}$$

Substituting eqn (D.10) into eqn (D.3), the equality now becomes

$$\sum_{n=1}^{\infty} a_n M_n'(y) = -\sum_{n=1}^{\infty} c_n O_n'(y) - \sum_{n=1}^{\infty} d_n P_n'(y) - W'(y) \tag{D.12}$$

with

$$\begin{aligned} M_n'(y) &= M_n(y) + \sum_{i=1}^{\infty} \gamma_{ni} N_i(y), \quad O_n'(y) = O_n(y) + \sum_{i=1}^{\infty} \delta_{ni}^{(1)} N_i(y), \\ P_n'(y) &= P_n(y) + \sum_{i=1}^{\infty} \delta_{ni}^{(2)} N_i(y), \quad W'(y) = W(y) + \sum_{i=1}^{\infty} \delta_i^{(0)} N_i(y). \end{aligned} \tag{D.13}$$

Using the same procedure, the orthonormal function $C_n(y)$ is constructed from $M_n'(y)$ as

$$C_n(y) = \sum_{i=1}^{n} (L_{in}/L_{nn}) M'_n(y) \tag{D.14}$$

where L_{in} is the cofactor of the element e_{in} of Δ_n, which is defined as

$$\Delta_n = \begin{vmatrix} e_{11} & e_{12} & \cdots & e_{in} \\ e_{21} & & & \vdots \\ \vdots & & & \vdots \\ e_{n1} & \cdots & \cdots & e_{nn} \end{vmatrix}, \quad e_{in} = \int_{y_1}^{y_2} M'_n(y) M'_i(y)\, dy. \tag{D.15}$$

Using eqns (D.12) and (D.14), coefficients a_n can be expressed by c_k and d_k as follows

$$a_n = \sum_{k=1}^{\infty} c_k \zeta_{nk}^{(1)} + \sum_{k=1}^{\infty} d_k \zeta_{nk}^{(2)} + \zeta_n^{(0)} \tag{D.16}$$

with

$$\zeta_{nk}^{(1)} = -\sum_{j=n}^{\infty} \{L_{nj}/(K_j L_{jj})\} \int_{y_1}^{y_2} O'_k(y) C_j(y)\, dy$$

$$\zeta_{nk}^{(2)} = -\sum_{j=n}^{\infty} \{L_{nj}/(K_j L_{jj})\} \int_{y_1}^{y_2} P'_k(y) C_j(y)\, dy$$

$$\zeta_n^{(0)} = -\sum_{j=n}^{\infty} \{L_{nj}/(K_j L_{jj})\} \int_{y_1}^{y_2} W'(y) C_j(y)\, dy$$

$$K_j = \int_{y_1}^{y_2} C_j^2(y)\, dy. \tag{D.17}$$

Substituting eqn (D.16) into eqn (D.10), we obtain the next relation

$$b_n = \sum_{k=1}^{\infty} c_k \varepsilon_{nk}^{(1)} + \sum_{k=1}^{\infty} d_k \varepsilon_{nk}^{(2)} + \varepsilon_n^{(0)} \tag{D.18}$$

with

$$\varepsilon_{nm}^{(1)} = \delta_{nk}^{(1)} + \sum_{i=1}^{\infty} \gamma_{ni} \zeta_{ik}^{(1)}, \quad \varepsilon_{nk}^{(2)} = \delta_{nk}^{(2)} + \sum_{i=1}^{\infty} \gamma_{ni} \zeta_{ik}^{(2)}$$

$$\varepsilon_n^{(0)} = \delta_n^{(0)} + \sum_{i=1}^{\infty} \gamma_{ni} \zeta_i^{(0)}. \tag{D.19}$$

Coefficients a_n and b_n can now be represented by c_n and d_n through eqns (D.16) and (D.18). Replacing coefficients a_n and b_n in eqns (D.1) and (D.2) with eqns (D.16) and (D.18), the equality becomes

$$\sum_{n=1}^{\infty} c_n G_n^*(x) + \sum_{n=1}^{\infty} d_n H_n^*(x) = -U^*(x) \tag{D.20}$$

$$\sum_{n=1}^{\infty} c_n K_n^*(x) + \sum_{n=1}^{\infty} d_n L_n^*(x) = -V^*(x) \quad \text{for } (x_1 \leqq x \leqq x_2) \tag{D.21}$$

with

$$G_n^*(x) = G_n(x) + \sum_{k=1}^{\infty} \{\zeta_{nk}^{(1)} E_k(x) + \varepsilon_{nk}^{(1)} F_k(x)\}$$

$$H_n^*(x) = H_n(x) + \sum_{k=1}^{\infty} \{\zeta_{nk}^{(2)} E_k(x) + \varepsilon_{nk}^{(2)} F_k(x)\}$$

$$K_n^*(x) = K_n(x) + \sum_{k=1}^{\infty} \{\zeta_{nk}^{(1)} I_k(x) + \varepsilon_{nk}^{(1)} J_k(x)\}$$

$$L_n^*(x) = L_n(x) + \sum_{k=1}^{\infty} \{\zeta_{nk}^{(2)} I_k(x) + \varepsilon_{nk}^{(2)} J_k(x)\}$$

$$U^*(x) = U(x) + \sum_{k=1}^{\infty} \{\zeta_k^{(0)} E_k(x) + \varepsilon_k^{(0)} F_k(x)\}$$

$$V^*(x) = V(x)+\sum_{k=1}^{\infty}\{\zeta_k^{(0)}I_k(x)+\varepsilon_k^{(0)}J_k(x)\}. \tag{D.22}$$

The same procedure is repeated. Letting $P_n^*(x)$ be a set of functions satisfying orthogonality condition

$$\int_{x_1}^{x_2} P_m^*(x)P_n^*(x)\,dx = N_n\,\delta_{mn},\quad N_n = \int_{x_1}^{x_2}\{P_n^*(x)\}^2\,dx \tag{D.24}$$

$P_n^*(x)$ can now be constructed from a set of functions $L_n^*(x)$ as

$$P_n^*(x) = \sum_{i=1}^{\infty}(M_{in}^*/M_{nn}^*)L_i^*(x) \tag{D.25}$$

where M_{in}^* is the coefactor of the element d_{in}^* of D_n^*, which is defined as

$$D_n^* = \begin{vmatrix} d_{11}^* & d_{12}^* & \dots & d_{1n}^* \\ d_{21}^* & & & \vdots \\ \vdots & & & \vdots \\ d_{n1}^* & \dots & \dots & d_{nn}^* \end{vmatrix},\quad d_{in}^* = \int_{x_1}^{x_2} L_i^*(x)L_n^*(x)\,dx. \tag{D.26}$$

Representing the second series in eqn (D.21) by the orthogonal series $P_n^*(x)$ with coefficients e_n^*, the following equations can now be given

$$\sum_{n=1}^{\infty} d_nL_n^*(x) = \sum_{n=1}^{\infty} e_n^*P_n^*(x) = -V^*(x)-\sum_{n=1}^{\infty} c_nK_n^*(x). \tag{D.27}$$

Using the second equality in eqn (D.27), coefficients e_n^* can be expressed as

$$e_n^* = -(1/N_n^*)\int_{x_1}^{x_2}\left\{V^*(x)+\sum_{n=1}^{\infty} c_iK_i^*(x)\right\}P_n^*(x)\,dx. \tag{D.28}$$

The first equality yields the following expression

$$d_n = \sum_{n=1}^{\infty}\gamma_{ni}^*c_i+\delta_n^* \tag{D.29}$$

with

$$\gamma_{ni}^* = -\sum_{j=1}^{\infty}\{M_{nj}^*/(N_j^*M_{jj}^*)\}\int_{x_1}^{x_2} K_i^*(x)P_j^*(x)\,dx$$

$$\delta_n^* = -\sum_{j=1}^{\infty}\{M_{nj}^*/(N_j^*M_{jj}^*)\}\int_{x_1}^{x_2} V^*(x)P_j^*(x)\,dx. \tag{D.30}$$

Substituting eqn (D.29) into eqn (D.20) reduces to

$$\sum_{n=1}^{\infty} c_nG_n^{**}(x) = -U^*(x)-\sum_{i=1}^{\infty}\delta_i^*H_i^*(x) \tag{D.31}$$

with

$$G_n^{**}(x) = G_n^*(x)+\sum_{i=1}^{\infty}\gamma_{ni}^*H_i^*(x). \tag{D.32}$$

Finally, coefficients c_n can be determined by

$$c_n = \sum_{j=n}^{\infty} q_j^*(S_{nj}^*/S_{jj}^*) \tag{D.33}$$

with

$$q_j^* = (-1/R_i^*)\int_{x_1}^{x_2} U^*(x)Q_j^*(x)\,\mathrm{d}x, \quad Q_j^*(x) = \sum_{i=1}^{\infty}(S_{ni}^*/S_{jj}^*)G_n^{**}(x) \tag{D.34}$$

where S_{jn}^* is the cofactor of the element e_{jn}^* of the determinant Δ_n^*

$$\Delta_n^* = \begin{vmatrix} e_{11}^* & e_{12}^* & \dots & e_{1n}^* \\ e_{21}^* & & & \vdots \\ \vdots & & & \vdots \\ e_{n1}^* & \dots & \dots & e_{nn}^* \end{vmatrix}, \quad e_{jn}^* = \int_{x_1}^{x_2} G_j^{**}(x)G_n^{**}(x)\,\mathrm{d}x, \tag{D.35}$$

and $Q_j^*(x)$ are the orthogonal functions that satisfy

$$\int_{x_1}^{x_2} Q_m^*(x)Q_n^*(x)\,\mathrm{d}x = R_n^*\delta_{mn}, \quad R_n^* = \int_{x_1}^{x_2}\{Q_n^*(x)\}^2\,\mathrm{d}x. \tag{D.36}$$

Coefficients c_n, d_n, a_n, b_n are calculated using eqn (D.33), (D.29), (D.16) and (D.18), respectively.